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Numerical solution for the linear transient heat conduction equation using an Explicit Green's Approach

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ABSTRACT

This paper presents a novel numerical solution algorithm for the linear transient heat conduction equation using the 'Explicit Green's Approach' (ExGA). The method uses the Green's matrix that represents the domain of the problem to be solved in terms of the physical properties and geometrical characteristic. The Green's matrix is the problem discrete Green's function determined numerically by the Finite Element Method (FEM). The ExGA allows explicit time marching with time step larger than the one required by FEM, without losing precision. The ExGA numerical results are quite accurate when compared to analytical solutions and to numerical solutions obtained by the FEM.

1. Introduction

Green's functions are very powerful tools for obtaining solutions to transient and steady-state linear heat conduction problems. They can also be applied to find solution of some convection problems and to model many other phenomena which are described by the same type of equation, i.e., those that involve solution of diffusion-type partial differential equations. A Green's function (GF) is a basic solution of a specific differential equation with homogeneous boundary conditions; for transient heat conduction, a GF describes the temperature caused by an instantaneous, local energy pulse. In addition to originating efficient numerical solution procedures, GF based methods also provide a better understanding of the nature of diffusion processes for heat conduction.

GFs have been used in the solution of equations of transient heat conduction for many decades, a classic text on the subject is the book by Carslaw and Jaeger [1], in which an introduction to the use of GFs based methods for heat conduction problems is presented. In that study they obtained the GFs through Laplace transforms. Other important reference concerning GFs is the book of Özişik [2] where the use and advantages of GFs based methods are described.

Morse and Feshbach [3], Butkovskiy [4], Greenberg [5], Roach [6], and Stakgold [7] also presented works describing GFs use for several applications. Beck [8] extended solutions presented by Özişik [2] by including the 'm^2T' term, which enters the governing equation in some heat problems. Approximate methods of finding GFs were developed by Haji-Sheikh and Lakshminarayanan [9] and Haji-Sheikh [10], who used the Galerkin-based integral method. Later, Beck et al. [11] presented an extensive compilation concerning GFs, containing a clear derivation of GF analytical solutions and a systematic and practical approach to the solution of diffusion-type problems. They derived 25 different GFs and analyzed in detail their multiplicative properties. Cole and McGahan [12] used a GF based approach to calculate the temperature of a multilayer medium heated by laser radiation. Feng and Michaelides [13] presented modified Green's functions (MGFs) to model heat transfer in a homogeneous or a composite solid body. Kuo and Chen [14], motivated by studies of Norris [15] and Martin et al. [16], elaborated further along this line of research, but focused on a simpler mathematical framework, the conduction phenomena, aiming at finding exact, closed-form expressions for Green's functions. Sutradhar et al. [17] worked with GFs for 3D transient heat conduction and graded materials, employing a Laplace transform boundary element method (LTBEM) approach, where the numerical implementation was performed using a Galerkin approximation. Gray et al. [18] worked with Green's functions and boundary integral analysis for exponentially graded materials in heat conduction.

Using GFs in heat conduction problems has several advantages. First, it is a powerful and flexible method, since the derived GFs for a given geometry may be used in conjunction with a variety of initial and boundary conditions; second, a systematic procedure is available for obtaining GFs, i.e., once these functions are obtained and tabulated, they may be used without any effort spent on the

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Nomenclature

$[C]$	Capacitance matrix	T_0	Temperature prescribed at $t = 0$
$\{F\}$	Thermal flux vector	$\{\dot{T}\}$	$\{\partial T/\partial t\}$
$[G]$	Green's matrix	t, τ	Time
g	Heat source	x	Horizontal Cartesian coordinate
$[I]$	Identity matrix		
$[K]$	Thermal conductance matrix	*Greek symbols*	
k	Thermal conductivity	α	Thermal diffusivity
q	Heat flux	Γ	Boundary of the domain
$\{T\}$	Vector of equivalent nodal temperature	Ω	Domain of the problem
T	Temperature		

details of their derivation; third, 1-D GFs may be used as building blocks to obtain 2- and 3-D solutions to suitable problems. The specifics of the multiplication process to obtain 2- and 3-D solutions are presented by Özişik [2]. A treatise by Greenberg [5] exposes the usefulness of the GF method.

Finding Green's functions corresponding to certain physical phenomena, possibly incorporated with suitably prescribed boundary data, is one of the fundamental subjects in mathematical physics. The knowledge of Green's functions can serve as a basic ingredient to construct the fields via superposition under distributed sources and general boundary data. The literature exhibits analytical expressions for GFs of a great variety of physical phenomena including wave propagation [19], anisotropic piezoelectricity, thermoelasticity, poroelasticity [15] among others; however, the majority of the existing fundamental solutions is under the condition that the material be homogeneous.

The field equations for graded media are in general governed by partial differential equations with position-dependent coefficients, and thus finding explicit solutions for this media GFs is quite complicated. Typical solutions of GFs for this case are often expressed as series or integral forms in a transformed space, which commonly limit their applications. For some aspects of applications, it is often desirable to have closed-form expressions for Green's functions, such as in effective medium theories and in boundary integral methods; however, in graded media this is not always possible. Success can be achieved for some cases, as illustrated in the recent study by Martin et al. [16] who derived the Green's function for a three-dimensional exponentially graded elastic solid, in which the Lamé constants vary exponentially in a certain direction, and in Sutradhar et al. [17] and Gray et al. [18], mentioned above.

The framework proposed in this article includes numerical solution for the linear transient heat conduction equation using the 'Explicit Green's Approach' (ExGA). This method allows the progress over time using the Green's matrix (GM), which represents the domain of the problem to be solved in terms of the physical and geometrical properties. The GM can be found without the knowledge of the analytical expression of the problem Green's function, which is, in many cases, difficult or even impossible of being worked out. However, in this paper, this matrix is determined numerically by the FEM, but it can also be found by other numerical methods such as: finite differences (FDM), finite volumes (FVM), etc.

The ExGA leads to an explicit time marching procedure with no time step restriction. More specifically, when sub-steps are used to calculate GMs, one can easily have, for instance, time steps 100 times larger (in fact there is no limit) than those of a typical FEM approach. Notice that the GM is just determined for the first interval of time, allowing its use for the next time steps. Moreover, the GM could be reused for other problems that have the same domain and physical characteristics.

Problems analyzed here with different initial and boundary conditions (Dirichlet and Neumann) show that the ExGA has a good performance when compared to analytical solutions found in the literature and to numerical solutions obtained by the FEM.

In the next section, a review and the mathematical background of the ExGA method using the FEM and the Laplace transform is presented. In Section 3, a numerical procedure to determine the GM is outlined. In Section 4, four examples are discussed. Finally, in the last section, some conclusions about this work are presented.

2. The Explicit Green's Approach (ExGA)

2.1. Review

With the objective of analyzing engineering problems efficiently, time-domain analysis plays a crucial role in scientific computations. There is a pressing need to develop effective computational algorithms and tools, especially for large-scale transient linear and nonlinear problems. At present, the so called direct time integration methods (DTIM), based either on physical or modal coordinates, are widely employed for a variety of engineering applications. Algorithms based on physical coordinates are quite popular in many commercial codes because of their various inherent advantages. However, for certain applications, modal coordinates continue to be the choice of some analysts, especially for linear systems, long time responses and repeated analyses. The basic factors worthy of consideration for computational algorithms for time-dependent problems are: the need for good accuracy while preserving the underlying physics, low CPU time and storage requirements, and easy implementation.

Analytical approaches, although strongly recommended to be pursued, are not feasible for some linear/nonlinear situations, especially, for complex geometries and large scale engineering computations. After the spatial discretization of transient field problems, there are many numerical approximation methods developed from various viewpoints that have been introduced for the time integration and solution of this class of problem. These include finite difference approximations for the time derivatives, which lead to the referred direct time integration one-step and multi-step methods (Belytschko and Hughes [21]) and formulations via a weighted residual approach, which have indeed provided certain useful generalizations to a limited extent (Wood [22]).

Employing finite elements as the principal analysis tool, the traditional practices and approaches for transient field problems first involve the semi-discretization of the partial differential equations to yield a set of ordinary differential equations in time. Finite difference approximations are then most customarily employed for the time derivative terms to obtain the necessary recursive time-stepping algorithmic relations. These algorithms are generally categorized as explicit and implicit.

Several different techniques of numerical analysis of transient nonlinear/linear thermal problems exist. Finite difference (FDM), finite element (FEM), finite volume (FVM), boundary element (BEM) methods are the most used techniques approaches for the spatial discretization. Within the context of the so-called semi-discretization process, which leads to a system of first-order ordinary differential equations in time, numerous time discretization approaches exist, which can be employed for evaluating the transient thermal response. Among these methods, the most common direct time integration algorithms are those belonging to the trapezoidal α-family (Hughes [23]). Whereas explicit solution techniques such as forward Euler ($\alpha = 0$) offer simplicity and are relatively easy to codify, they are only first-order time accurate and conditionally stable, thus imposing a severe time-step restriction. On the other hand, implicit solution techniques such as the Crank–Nicolson method ($\alpha = 0.5$) are unconditionally stable and do not impose such severe restrictions; this is the only method of the trapezoidal family that is second-order time accurate.

The literature contains many classical approaches for time marching; for a complete review see Tamma et al. [24]. Explicit procedures are preferable in a large class of problems because they are computationally cheaper and faster (Baumeister and Kreider [25], Su and Tabarrok [26]). Besides the works previously mentioned, Hulbert [27] presented a second-order accurate explicit subcycling algorithm, and Mohtar and Segerlind [28] utilized a dynamic time-step with forward, central and backward differences and Galerkin schemes. On the other hand, implicit schemes have been developed in order to seek high-order accurate approaches (Zhang [29] and Karaa [30]).

Mansur et al. [31] presented a family of algorithms to time integrate hyperbolic partial differential equations based on numerical GM; the approach presented was denominated ExGA. These authors presented a discussion where they associate the ExGA approach with the classical time-domain boundary integral equation (Mansur [19]) method. It is worth mentioning that the term 'explicit' is employed in the sense that Green's functions are explicitly computed.

Wrobel [32] worked out step-by-step Green's functions based algorithms using analytical fundamental solutions of homogeneous media. Soares and Mansur [33] developed a formulation to compute Green's functions implicitly by the Newmark time marching scheme. Zhong and Williams [34] derived a new family of unconditionally explicit or implicit algorithms based on analytical solution of first-order ordinary differential equations in which the concept of Green's functions is also implicitly presented.

Fung [35] dealt with numerical determination of Green's functions by alternative formulations applied to MDOF spring-mass systems. Loureiro [36] and Dors [37] presented the general approach and defined in which class of problems the new approaches are recommended instead of the classical ones.

As Mansur et al. [31], Loureiro [36] and Dors [37] did for the scalar wave equation, the time-domain boundary integral equation presented by Mansur [19] and Wrobel [32] can be used to explain concepts of the developments presented here for the heat conduction concerning the ExGA. In the formulation discussed in the present paper, the major limitation of BEM approaches is removed, i.e., it is not necessary to have an analytical expression for the Green's function of a problem; rather, it is computed numerically. Any standard numerical method can be employed to compute the Green's function of the problem, thus there is no limitation at all. For example, the medium can be nonhomogeneous, anisotropic, visco-elastic, poroelastic, etc. The price to pay for the aforementioned generality is the discretization of the domain; however, substantial accuracy and stability improvement are achieved. In fact, if sub-steps are employed, one is free to choose the maximum time-step for which explicit Green's function based algorithms are stable and accurate. As mentioned above one can easily employ time steps thousands of times larger than those permitted by the central difference or Runge–Kutta methods. The only restriction on the time-step length is now concerned with having a good picture of the time response history.

2.2. The ExGA method

The present section describes the Explicit Green's Approach for integrating the heat conduction equation. The algorithm is based on the calculation of the Green's function in nodal coordinates by the finite element method.

The spatial discretization, using the finite element method (FEM), of the parabolic non-homogeneous heat conduction equation can be represented in matrix form as (Hughes [23]):

$$[C]\{\dot{T}(t)\} + [K]\{T(t)\} = \{F(t)\} \tag{1}$$

with the initial condition,

$$\{T(0)\} = \{T\}^0 \tag{2}$$

where $[C]$ is the capacitance matrix, $[K]$ is the thermal conductance matrix, and $\{F\}$ is a vector of equivalent nodal heat loads. These matrices and the vector above are calculated by the following expressions:

$$C_{ij} = \sum_{e=1}^{n} \frac{k}{\alpha} \iint_{\Omega_e} N_i N_j \mathrm{d}\Omega \tag{3}$$

$$K_{ij} = \sum_{e=1}^{n} k \iint_{\Omega_e} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) \mathrm{d}\Omega \tag{4}$$

$$F_i = \sum_{e=1}^{n_c} \int_{\Gamma} \bar{q} N_i \mathrm{d}\Gamma - \sum_{e=1}^{n_f} \iint_{\Omega e} g N_i \mathrm{d}\Omega \tag{5}$$

where $i, j = 1, \ldots, n_n$ (number of nodes), N_i and N_j are the shape functions, n is the number of elements in the domain, n_c is the number of elements with face(s) on the boundary which have prescribed heat flux, and n_f is the number of elements of the region where a source is being applied. Quadrilateral finite elements have been used in this work.

Applying the Laplace transform $\mathscr{L}$ to Eq. (1) gives (see Boyce and DiPrima [38]):

$$\mathscr{L}([K]\{T(t)\} + [C]\{\dot{T}(t)\}) = \mathscr{L}(\{F(t)\}) \tag{6}$$

Using properties of the Laplace transform, one has:

$$[K]\mathscr{L}(\{T(t)\}) + [C]\mathscr{L}(\{\dot{T}(t)\}) = \mathscr{L}(\{F(t)\}) \tag{7}$$

$$[K]\mathscr{L}(\{T(t)\}) + [C][s\mathscr{L}(\{T(t)\}) - \{T(0)\}] = \mathscr{L}(\{F(t)\}) \tag{8}$$

$$([K] + s[C])\mathscr{L}(\{T(t)\}) = [C]\{T(0)\} + \mathscr{L}(\{F(t)\}) \tag{9}$$

Therefore, the temperature vector in the Laplace domain can be written as:

$$\mathscr{L}(\{T(t)\}) = H(s)([C]\{T(0)\} + \mathscr{L}(\{F(t)\})) \tag{10}$$

where

$$[H(s)] = ([K] + s[C])^{-1} \tag{11}$$

represents a transfer function. Finally, applying the Laplace inverse transform to Eq. (10) and considering Eq. (11), one can obtain the expression that represents the temperature vector in the time domain, as follows:

$$\{T(t)\} = \mathscr{L}^{-1}([H(s)])[C]\{T(0)\} + \mathscr{L}^{-1}(H(s)\{F(s)\}) \tag{12}$$

Observe that if $[H(s)]$ is the transfer function in the Laplace domain, then its inverse transform is the transfer function in the time domain, which is the time domain Green's function in matrix form. Thus,

$$[G(t)] = \mathscr{L}^{-1}([H(s)]) \tag{13}$$

Therefore, the final expression of the vector temperature in terms of the Green's matrix is:

$$\{T(t)\} = [G(t)][C]\{T(0)\} + \int_0^t [G(t-\tau)]\{F(\tau)\}d\tau \tag{14}$$

3. Numerical procedure

3.1. Green's matrix

Green's function (GF) is usually called impulsive response of the system, as it is explained in Özişik [2, chapter 6]. It can be obtained numerically from Eq. (1), considering homogeneous boundary conditions and an impulsive heat source applied at a source point, which coincides with a node of the space mesh. Then, the Green's matrix can be obtained by the solution of the following system of ordinary differential equations in the time:

$$[C][\dot{G}(t)] + [K][G(t)] = [I]\delta(t) \tag{15}$$

with the initial condition,

$$[G(0)] = 0 \tag{16}$$

where $[I]$ represents the identity matrix and $\delta(t)$ is the Dirac delta function.

The problem described by Eq. (15) is equivalent to the homogeneous initial value problem with application of a certain initial condition (see Özişik [20]). In the present work, it was verified that the correct equivalence is given by the problem presented below:

$$[C][G(t)] + [K][G(t)] = 0 \tag{17}$$

with the initial condition,

$$[G(0)] = [C]^{-1} \tag{18}$$

Here the Green's matrix is computed implicitly through the range $[0, \Delta t]$ with the Crank–Nicolson scheme. It is worth to point out that the new method becomes more stable and accurate when a substep procedure is adopted to obtain the Green's matrix. If the capacitance matrix is lumped (diagonal), computing its inverse is fairly cheap.

3.2. Convolution integral

In this section, it is presented a scheme to perform the numerical calculus of the convolution integral given by the second term on right-hand side of Eq. (14). Assuming that the time step is Δt the temperature solution vector at any time can be evaluated recursively as:

$$\{T\}^{t+\Delta t} = [G]^{\Delta t}[C]\{T\}^t + \int_t^{t+\Delta t} [G(t+\Delta t-\tau)]\{F(t+\tau)\}d\tau \tag{19}$$

The second term on the right-hand side of Eq. (19) can be rewritten as:

$$\int_0^{\Delta t} G(\Delta t-\tau)F(\tau)\mathrm{d}\tau \tag{20}$$

The functions $\{F(\tau)\}$ and $\tau(\xi)$ are here interpolated in the interval $[0, \Delta t]$ as indicated below,

$$F(\tau(\xi)) = F_i N_1(\xi) + F_{i+1}N_2(\xi) \quad \text{and} \quad \xi \in [-1, 1] \tag{21}$$

$$\tau(\xi) = t_i N_1(\xi) + t_{i+1}N_2(\xi) \quad \text{and} \quad \xi \in [-1, 1] \tag{22}$$

where N_1 and N_2 are linear interpolation functions given by:

$$N_1(\xi) = \frac{1}{2}(1-\xi) \quad \text{and} \quad N_2(\xi) = \frac{1}{2}(1+\xi) \tag{23}$$

After mathematical manipulations, one can write $(t_{i+1} - \tau)$ as a function of N_1 and Δt, and $d\tau$ as a function of $d\xi$ and Δt as indicated below:

$$t_{i+1} - \tau = \Delta t N_1(\xi) \tag{24}$$

$$\mathrm{d}\tau = \frac{\Delta t}{2}\mathrm{d}\xi \tag{25}$$

Applying the expressions Eqs. (21), (24) and (25) in Eq. (20), one obtains the final expression, which is ready for computational implementation:

$$\int_{t_i}^{t_{i+1}} G(t_{i+1}-\tau)F(\tau)\mathrm{d}\tau = \frac{\Delta t}{2}\left\{\left[\int_{-1}^{1} G(\Delta t N_1(\xi))N_1(\xi)\mathrm{d}\xi\right]F_i + \left[\int_{-1}^{1} G(\Delta t N_1(\xi))N_2(\xi)\mathrm{d}\xi\right]F_{i+1}\right\} \tag{26}$$

In the present work the integrals of Eq. (26) are calculated by Newton–Cotes approximations with interpolation polynomials of order one (trapezoidal rule).

4. Examples

In this section, four examples are presented to illustrate the methodology employed in this work. The objectives of the examples are: (i) to solve a problem of heat conduction, without sources and with null flux at the boundary, so that the answer for the temperature field is given just by the first part of the right-hand side of Eq. (19), that is, eliminating the contribution of the convolution integral; (ii) to simulate one-dimensional situations to test the Dirichlet and Neumann boundary conditions; and (iii) to analyze the two-dimensional situation with a thermal shock on the boundary, so that the answer for the temperature field is given by the complete Eq.(19).

A thin rectangular plate of sides $a = 10.0$ m and $b = 8.0$ m was considered for examples 1 and 4. Examples 2 and 3, that represent the one-dimensional problem, considered the side $b = 1.0$ m to reduce computer run time. The adopted physical coefficients were $\alpha = 1.0\ \mathrm{m^2/s}$, $k = 1.0\ \mathrm{W/^\circ C\ m}$. The Green's matrix was determined for FEM using square elements (see Figs. 1 and 2) for the spatial discretization and the finite difference method with the Crank–Nicolson scheme for time integration. The ExGA method was compared to the FEM with the same spatial discretization

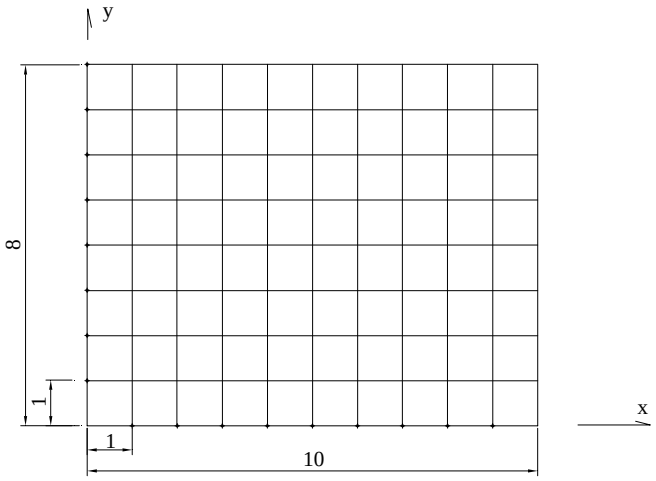

Fig. 1. Geometry and mesh of examples 1 and 4.

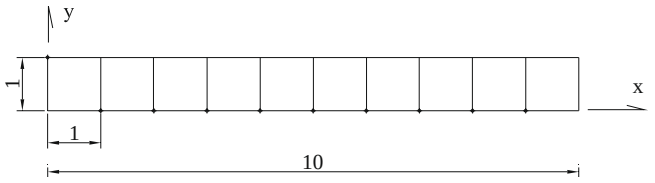

Fig. 2. Geometry and mesh of examples 2 and 3.

and with the same implicit scheme in the time discretization used in the calculus of the Green's matrix.

4.1. Example 1

This example considers a rectangular domain with zero thermal flux on all sides (see Fig. 3), with initial condition $T(x,y,0) = \delta(x,y|5,4)$, were (5,4) is the central point of domain. The Green's matrix was calculated using a time step $\Delta t = 0.1$ s and the meshes and sub-steps division shown in Table 1; the coarsest mesh of 80 elements is that shown in Fig. 1.

The analytical Green's function for this two-dimensional problem is given by Beck et al. [11, p. 503] . The solution, true for small values of $\alpha(t-\tau)/a^2$ and $\alpha(t-\tau)/b^2$ (both $\leqslant 0.022$), is given by:

$$G(x,y,t|x',y',\tau) = \frac{1}{4\pi\alpha(t-\tau)}\left\{\sum_{i=1}^{9}\exp\left[-\frac{r_i^2}{4\alpha(t-\tau)}\right]\right\} \tag{27}$$

where

$$\begin{aligned}
r_1^2 &= (x-x')^2 + (y-y')^2 \\
r_2^2 &= (x-x')^2 + (y+y')^2 \\
r_3^2 &= (x-x')^2 + (2b-y-y')^2 \\
r_4^2 &= (x+x')^2 + (y-y')^2 \\
r_5^2 &= (x+x')^2 + (y+y')^2 \\
r_6^2 &= (x+x')^2 + (2b-y-y')^2 \\
r_7^2 &= (2a-x-x')^2 + (y-y')^2 \\
r_8^2 &= (2a-x-x')^2 + (y+y')^2 \\
r_9^2 &= (2a-x-x')^2 + (2b-y-y')^2
\end{aligned} \tag{28}$$

The graphic of Fig. 4 displays ExGA method results when the mesh is refined, maintaining the time step constant and increasing the number of sub-steps. These results are the same found by FEM when the time step used is equal to that utilized by ExGA divided by the number of sub-steps (*ss*), i.e., $(\Delta t)_{FEM} = (\Delta t)_{ExGA}/ss$. It is important to observe that the differences between the graphs of Fig. 4 and the analytical solution are not related to convergence,

Table 1
Number of elements and sub-steps, element side length and central point initial temperature for example 1

Elements	Sub-steps	Element side length (m)	$T(5,4,0)$ (°C)
80	10	1	1
320	20	0.5	4
1280	100	0.25	16

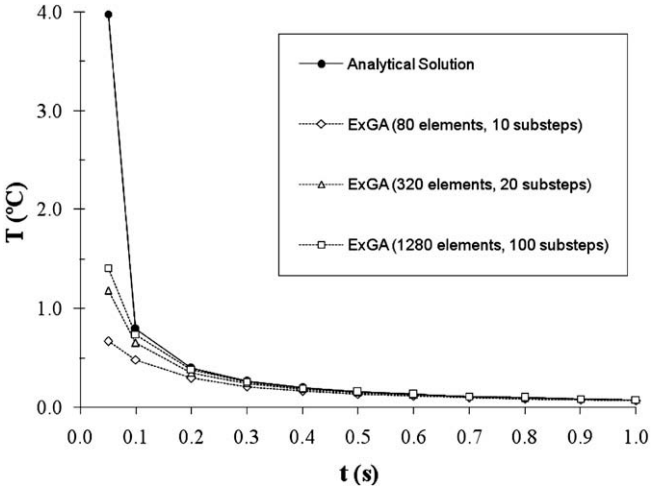

Fig. 4. Temperature at the central point (5,4) of the domain of example 1.

rather, it is because analytical Green's functions consider punctual loads whereas numerical ones consider a distribution over the mesh in the neighborhood of the source node. Thus, as demonstrated in Fig. 4, the more refined the mesh is the closer analytical and numerical solutions are. Also, it can be seen that as expected, when t tends to zero the analytical solution at the source point tends to infinite.

4.2. Example 2

The initial condition is $T(x,y,0) = 0.0$ °C throughout the domain. The boundary conditions are zero thermal flux on the two horizontal sides of the rectangular region with $T(0,y,t) = 0.0$ °C on the left vertical side and $T(a,y,t) = 1.0$ °C on the right vertical side (see Fig. 5). The time history of the temperature at the central point illustrated by the graph of Fig. 6, demonstrates the accuracy of the ExGA method when compared to the analytical solution and to the numerical solution by FEM. The time step used for the ExGA method was $\Delta t = 10.0$ s with 20 sub-steps to calculate the Green's

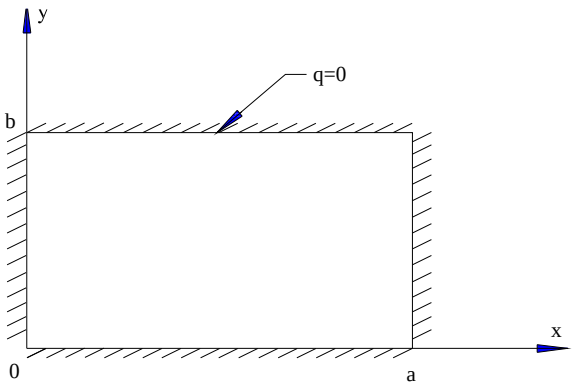

Fig. 3. Geometry and boundary conditions of example 1.

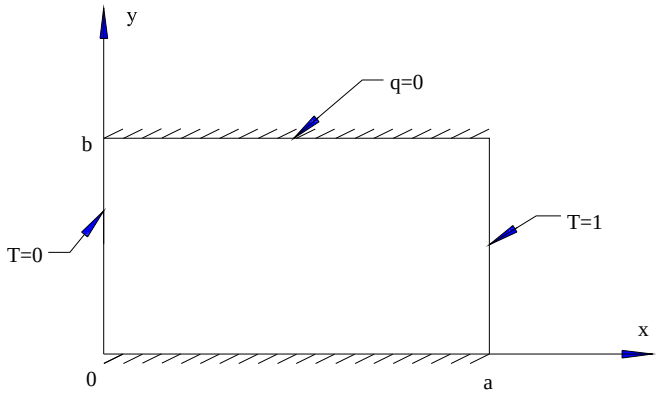

Fig. 5. Geometry and boundary conditions of example 2.

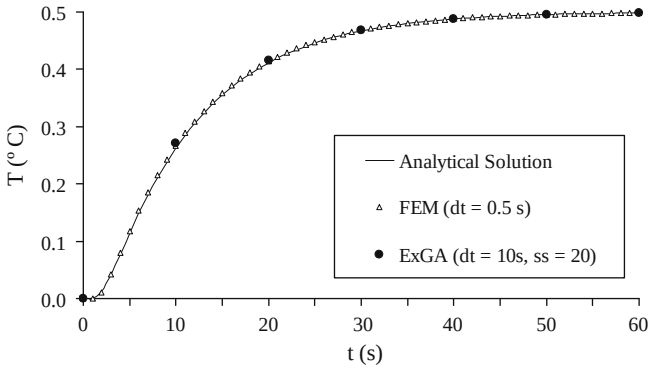

Fig. 6. Temperature at the central point of the domain of example 2.

matrix, while $\Delta t = 0.5$ s was adopted for the FEM. The analytical solution presented by Carslaw and Jaeger [1, p. 100] , with the last term equal zero because the initial condition (T_0) is equal zero, reads:

$$T(x,t) = T_1 + (T_2 - T_1)\frac{x}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{T_2\cos(n\pi) - T_1}{n} \times \sin\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-\alpha n^2\pi^2 t}{L^2}\right) \quad (29)$$

where T_1 and T_2 are the boundary conditions, and L is the length of the domain in the flux direction, which in this example is in the x direction.

4.3. Example 3

As it was considered in example 2, the initial condition is $T(x,y,0) = 0.0$ °C throughout the domain. The boundary conditions are: $q(x,0,t) = q(x,b,t) = 0.0$ W/m, $T(0,y,t) = 0.0$ °C and $q(a,y,t) = 0.1$ W/m (see Fig. 7). The time step used in the ExGA method was $\Delta t = 50.0$ s with 10 sub-steps to calculate the Green's matrix while $\Delta t = 5.0$ s was the time step for the FEM.

Time history of the temperature at the central point obtained with the ExGA method and the FEM are illustrated by the graph of Fig. 8. The one-dimensional analytical results for the temperature field indicated in Fig. 8 can be found in Carslaw and Jaeger [1, p. 113], and are given by:

$$T(x,t) = \frac{q_0 x}{k} - \frac{8q_0 L}{k\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \times \exp\left[\frac{-\alpha(2n+1)^2\pi^2 t}{4L^2}\right]\sin\left[\frac{(2n+1)\pi x}{2L}\right] \quad (30)$$

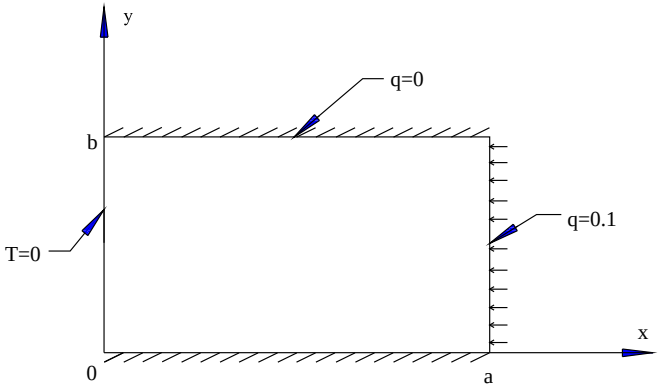

Fig. 7. Geometry and boundary conditions of example 3.

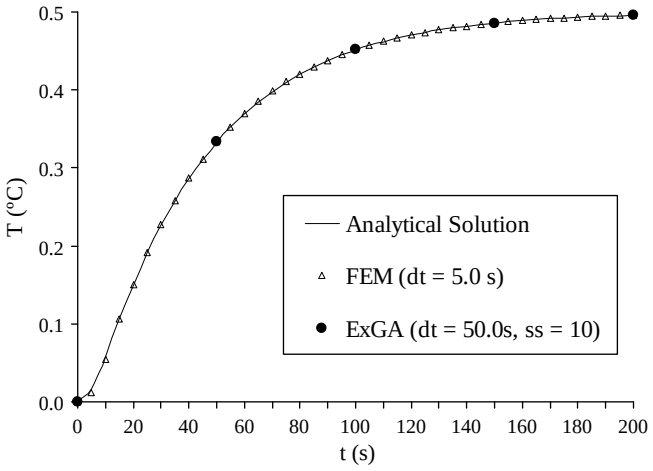

Fig. 8. Temperature at the central point of domain of example 3.

where q_0 is the Neumann boundary condition, and L is the length of the domain in the flux direction, which in this example is parallel to the x-axis.

4.4. Example 4

The initial condition is $T(x,y,0) = 0.0$ °C throughout the domain. The boundary conditions are unitary temperature T_1 on all sides of the region as shown in Fig. 9. The time step used by the ExGA method was $\Delta t = 5.0$ s with 10 sub-steps to calculate the Green's matrix, while $\Delta t = 0.5$ s was used by the FEM. Simulations with ExGA method and FEM were also performed and compared to the analytical solution presented by Carslaw and Jaeger [1, p. 185], adapted to 2-D, as given by:

$$T(x,y,t) = T_1 - \frac{16T_1}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{(2n+1)(2m+1)} \times \cos\left[\frac{(2n+1)\pi x}{2a}\right] \times \cos\left[\frac{(2m+1)\pi y}{2b}\right]\exp\left\{\frac{-\alpha\pi^2 t}{4}\left[\frac{(2n+1)^2}{a^2} + \frac{(2m+1)^2}{b^2}\right]\right\} \quad (31)$$

The numerical and the analytical solution results at the central point are depicted by graphs indicated in Fig. 10.

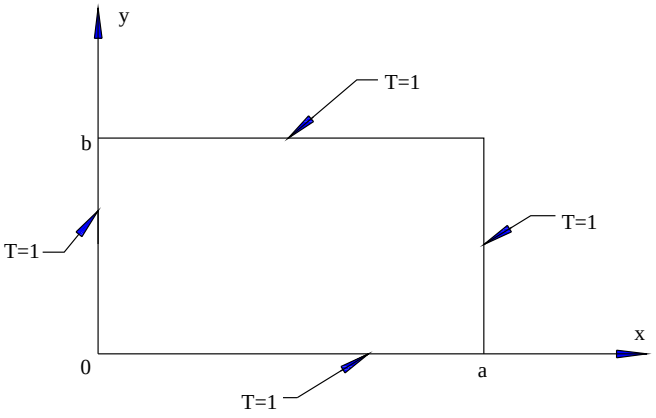

Fig. 9. Geometry and boundary conditions of example 4.

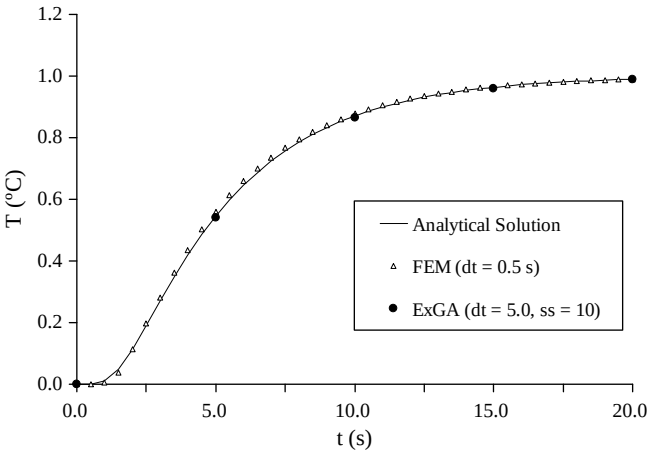

Fig. 10. Temperature at the central point of domain of example 4.

4.5. Discussion of results

The result of example 1 displays the expected behavior of a robust numerical method. The Green's function representation is good for regions not too close to the source point or else for time not too close to the initial time when the impulse is delivered. Correct representation of singularities by FEM, FDM, FVM and ExGA requires the development of special algorithms which is out of the scope of the present work. When the mesh was refined, the region over which the initial condition is applied became smaller, thus the numerical Green's function became closer to the analytical one. In addition, mesh refinement leads to a more accurate numerical result for the Green's matrix, and consequently for the temperature field. The examples 2 and 3 illustrate a one-dimensional heat conduction situation through typical problems of the literature for first and second kind boundary conditions. The example 4 simulates a two-dimensional heat conduction case, by means of a thermal shock on the boundary. In all the studied cases, the ExGA method provided accurate results and converged to the correct answer with refinements that do not demand great computational efforts.

5. Conclusions

This work presented a new approach for the solution of linear transient heat conduction problems. An important contribution of this article is the demonstration of the inverse capacitance matrix as the equivalent initial condition to the unitary pulse given by the Dirac delta function. If this was not possible, less accurate numerical Green's function would be obtained, and the ExGA method would not be competitive with the classic ones.

In the ExGA method, the temperature field is calculated explicitly on time through the Green's matrix (GM) which is determined by FEM using an implicit algorithm. The use of this implicit algorithm with sub-steps to compute the GM increases the computational cost; however, this apparent disadvantage becomes a positive aspect since the unconditional stability property inherent to implicit methods, such as the Crank–Nicolson scheme, is transmitted through the GM to the ExGA method.

Thus, with sub-steps not too small, one obtains an explicit algorithm which for any practical purpose can be considered unconditionally stable. In the examples 2, 3 and 4, a time step 20 times bigger than that of the FEM was considered. In fact using sub-steps allow the time-step 'length' to be as large as one wishes; the only limitation being a good representation of the time response picture. The accuracy of the ExGA algorithm which is quite good can be improved by considering higher-order polynomials to interpolate fluxes and in the Newton–Cotes algorithm.

The ExGA algorithm can be very suitable to multiple cases where only the load conditions (boundary or source) change while domain and physical properties remain unchanged, as in this case is possible to use the same Green's matrix for all analyses.

Finally, it is important to note that the method is new, thus, there are quite a large number of applications yet to be carried out.

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